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2001 J. Phys. A: Math. Gen. 34 491

(http://iopscience.iop.org/0305-4470/34/3/312)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 491–512 www.iop.org/Journals/ja PII: S0305-4470(01)14371-4

Partial Lie-point symmetries of differential equations

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Received 26 May 2000, in final form 30 October 2000

Abstract

When we consider a differential equation $\Delta = 0$ whose set of solutions is $\mathcal{S}_\Delta$, a Lie-point exact symmetry of this is a Lie-point invertible transformation T such that $T(\mathcal{S}_\Delta) = \mathcal{S}_\Delta$, i.e. such that any solution to $\Delta = 0$ is transformed into a (generally, different) solution to the same equation; here we define *partial* symmetries of $\Delta = 0$ as Lie-point invertible transformations T such that there is a non-empty subset $\mathcal{P} \subset \mathcal{S}_\Delta$ such that $T(\mathcal{P}) = \mathcal{P}$, i.e. such that there is a subset of solutions to $\Delta = 0$ which are transformed into one another. We discuss how to determine both partial symmetries and the invariant set $\mathcal{P} \subset \mathcal{S}_\Delta$, and show that our procedure is effective by means of concrete examples. We also discuss relations with conditional symmetries, and how our discussion applies to the special case of dynamical systems. Our discussion will focus on continuous Lie-point partial symmetries, but our approach would also be suitable for more general classes of transformations; the discussion is indeed extended to partial generalized (or Lie–Bäcklund) symmetries along the same lines, and in the appendix we will discuss the case of discrete partial symmetries.

PACS number: 0220T

Introduction

Symmetries are most useful in the study of differential equations [4, 13, 19, 24, 30, 31, 34], and they are useful in different ways.

On the one hand, one can consider symmetry reduction of differential equations and thus obtain classes of exact solutions; on the other hand, by definition, a symmetry of a differential equation transforms solutions into solutions, and thus symmetries can be used to generate new solutions from known ones.

We note that these transformations can yield highly non-trivial solutions starting from very simple known ones; the best example is possibly that provided by the symmetry transformation taking the trivial constant solution of the heat equation into the fundamental (Gaussian) one, as discussed in section 2.4 of [19].

Transformations which act point-like in the space of independent and dependent variables are also called Lie-point symmetries, to be distinguished from transformations of a more general nature, involving, for example, derivatives or integrals of these variables [19].

Several generalizations of standard Lie-point symmetries have been considered in the literature; we mention, in particular, the *conditional symmetries*, first introduced by Bluman and Cole [2, 3] and then systematized and applied by Levi and Winternitz [18, 32–34], i.e. transformations T which are *not* symmetries of the given differential equation, but such that this has some solution u_0 invariant under them, $T[u_0] = u_0$. The trivial example of this phenomenon is provided by non-symmetric equations admitting the zero solution: this obviously has a very large group of symmetries.

Much less trivial—and actually very useful—examples are considered in the literature; we refer, for example, to [34] for both a general discussion of the method and for relevant examples; this reference also lists a number of works where conditional symmetries have been applied to equations of physical relevance.

It should be noted that, in a way, conditional symmetries focus on (an extension of) the first use of symmetries mentioned above, i.e. on the search for invariant solutions.

One should also mention that conditional symmetries are strictly related to the so-called 'direct method' of Clarkson and Kruskal [9]; the group-theoretical understanding of this [18] involves conditional symmetries and is related to the 'non-classical method' of Bluman and Cole [2, 3] and to the 'side conditions' of Olver and Rosenau [20]. For the relation between conditional symmetries and the other mentioned approaches, see also the discussion by Pucci and Saccomandi [21] (and their recent paper [22]). For a related approach, see also the work of Fushchich and collaborators [12]. In [10], a symmetry formulation was used to transform partial differential equations (PDEs) on a bounded domain of $\mathbb{R}^n$ with boundary conditions into PDEs on a boundaryless manifold, the role of boundary conditions now being played by an extra symmetry condition; this approach has been applied, for example, in [11].

In this paper, we propose an extension of the notion of symmetries which goes in the direction of the other use, i.e. of transformations taking solutions to solutions. More specifically, let $\mathcal{S}_\Delta$ be the set of all solutions to a given differential equation $\Delta = 0$; an exact symmetry will be a transformation T which induces an action (in the suitable function space[3]) transforming each element $s \in \mathcal{S}_\Delta$ into a (generally, different) element $s' \in \mathcal{S}_\Delta$.

We will consider *partial symmetries*: these will be transformations T which induce an action (in the suitable function space) transforming each element $u \in \mathcal{P} \subset \mathcal{S}_\Delta$ into a (generally, different) element $u' \in \mathcal{P} \subset \mathcal{S}_\Delta$, for $\mathcal{P}$ some non-empty subset of $\mathcal{S}_\Delta$ (a more precise definition will be given below). Note that when $\mathcal{P} = \mathcal{S}_\Delta$ we actually have standard exact symmetries, while when $\mathcal{P}$ reduces to a single solution, or to a set of solutions each of them invariant under T, we recover the setting of conditional symmetries. Indeed, as we discuss in section 2, conditional symmetries will always be a special class of partial symmetries.

The reader could like to see immediately a simple example in order to better grasp this qualitative definition; here is one in terms of an infinitesimal symmetry generator, for a PDE for $u = u(x, y)$.

Example. Consider the vector field $X = (\partial/\partial x)$, which generates the translations of the variable x, and is an exact symmetry for any equation not depending explicitly on x. An equation, for instance, like $x(u_x - yu_y) + u_{xy} - u_y + yu_{yy}^2 = 0$ certainly does not admit translational symmetries $\partial/\partial x$ or $\partial/\partial y$; however, it admits the particular family of solutions $u = y\exp(x + \lambda)$ which are transformed into one another under the translations of the variable

[3] A more geometrical definition is also possible: the transformation, acting in the space of independent and dependent variables, has to map the graphs of solutions to graphs of, generally different, solutions (see [19]).

x, as this can be reabsorbed by a corresponding shift in the parameter λ. Note that none of these solutions is invariant under the x translation.

It should be mentioned that this notion of 'partial symmetries' had been considered in general terms by several authors; however, to the best of our knowledge, this has never gone beyond the stage of introducing an abstract notion, without a discussion of methods to implement this—nor *a fortiori* concrete applications—except for conditional symmetries, i.e. invariant solutions.

Indeed, the notion of partial symmetries also appears in early general definitions of conditional symmetries (the method actually being implemented, however, only in the stricter sense mentioned above, the notion of conditional symmetries passed to be used in the present sense). For further details we refer the reader to the works by Vorob'ev [25, 26], who also suggests a solution to some priority question by mentioning that this notion had already be considered by Klein [26].

In the following we will characterize partial symmetries in an operational way, and show—in theoretical terms but also by means of concrete examples—how they can be used to obtain solutions to nonlinear differential equations. We will mainly focus, as is customary in the symmetry study of differential equations, on continuous transformations, and actually study their infinitesimal generators (the reason being, as usual in this field, that the equations we obtain for these are much more palatable than those obtained by considering finite transformations); discrete transformations will be briefly considered in the appendix.

It should be stressed that, as also happens for conditional symmetries (but contrary to the case of exact Lie-point symmetries), the determining equations for (infinitesimal generators of) partial symmetries will be nonlinear; in general, we will thus be unable to determine all the partial symmetries to a given equation; thus we will need to have some hint, maybe on the basis of physical considerations, of what the partial symmetries could be for the method to be applicable with reasonable effort. Nevertheless, determination of one or some partial symmetries can already be of use in the search for exact solutions (as also happens for exact symmetries).

Finally, we note that we focus our discussion on Lie-point symmetries, but the definition and discussion can be extended to generalized symmetries (sometimes also called Lie–Bäcklund symmetries). We will also compare partial generalized symmetries with conditional generalized symmetries (for these, see [35]).

To conclude this introduction, we stress that the partial symmetry method can lead to a class of solutions including solutions which cannot be obtained either by exact symmetries or by conditional symmetries. The examples below show that this is actually the case.

The paper is organized as follows. In section 1 we introduce and define partial symmetries in a constructive way; i.e. also identifying the subset $\mathcal{P}$ of solutions which is left globally invariant by the partial symmetry and explaining how to compute this and the symmetry itself. In section 2 we discuss the relation between these partial symmetries and the conditional symmetries of Levi and Winternitz, and with the conditional generalized symmetries of Zhdanov. In section 3 we discuss how, in certain circumstances, the given partial symmetries guarantee the considered differential equation enjoying a 'partial superposition principle' (defined there). In section 4 we specialize our discussion to the case of dynamical systems. The final sections are devoted to a detailed discussion of concrete examples: these deal with PDEs of interest for physics in section 5, and with dynamical systems in section 6. As already mentioned, our discussion is at the level of infinitesimal symmetry generators and thus continuous partial symmetries, but in the appendix we briefly discuss discrete partial symmetries.

1. Partial symmetries of differential problems

Let us consider a general differential problem, given in the form of a system of ℓ differential equations, and briefly denoted, as usual, by

$$\Delta := \Delta(x, u^{(m)}) = 0 \tag{1.1}$$

where $\Delta := (\Delta_1, \Delta_2, \ldots, \Delta_\ell)$ are smooth functions involving p independent variables $x := (x_1, \ldots, x_p) \in \mathbb{R}^p$ and q dependent 'unknown' variables $u := (u_1, \ldots, u_q) \in \mathbb{R}^q$, together with the derivatives of the u_α with respect to the x_i ($\alpha = 1, \ldots, q;\ i = 1, \ldots, p$) up to some order m.

Let

$$X = \xi_i \frac{\partial}{\partial x_i} + \varphi_\alpha \frac{\partial}{\partial u_\alpha} \qquad \xi_i = \xi_i(x, u) \quad \varphi_\alpha = \varphi_\alpha(x, u) \tag{1.2}$$

be a given vector field, where ξ_i and φ_α are $p+q$ smooth functions. For notational simplicity, we will briefly denote by X^* the 'suitable' prolongation of X, i.e. the prolongation which is needed when one has to consider its application to the differential problem under consideration. Alternatively, we may consider X^* as the infinite prolongation of X; indeed, it is clear that only a finite number of terms are required and will appear in all the actual computations. As is well known [19], the vector field X is (the Lie generator of) an exact symmetry of the differential problem (1.1) if and only if[4]

$$X^*\Delta\big|_{\Delta=0} = 0 \tag{1.3}$$

i.e. if and only if the prolongation X^* (here obviously, $X^* = \mathtt{pr}^{(m)}(X)$, the mth prolongation of X) applied to the differential operator Δ defined by (1.1) vanishes once restricted to the set $S^{(0)} := \mathcal{S}_\Delta$ of the solutions to the problem $\Delta = 0$.

We now assume that the vector field X is *not* a symmetry of (1.1), hence $X^*\Delta\big|_{S^{(0)}} \neq 0$: let us put

$$\Delta^{(1)} := X^*\Delta. \tag{1.4}$$

This defines a differential operator $\Delta^{(1)}$, of order m' not greater than the order m of the initial operator Δ. Assume now that the set of simultaneous solutions of the two problems $\Delta = 0$ and $\Delta^{(1)} = 0$ is not empty, and let us denote by $S^{(1)}$ the set of these solutions. It can happen (see examples 2–5) that this set is mapped into itself by the transformations generated by X: *this situation is characterized precisely by the property*

$$X^*\Delta^{(1)}\big|_{S^{(1)}} = 0. \tag{1.5}$$

Then, in this case, we can conclude that, although X is not a symmetry for the full problem (1.1), it generates anyway a transformation which leaves globally invariant a family of solutions of (1.1): this family is precisely $S^{(1)}$.

However, it can also happen that $X^*\Delta^{(1)}|_{S^{(1)}} \neq 0$ (see examples 1 and 2), we then put

$$\Delta^{(2)} := X^*\Delta^{(1)} \tag{1.6}$$

and look for the solutions of the system

$$\Delta = \Delta^{(1)} = \Delta^{(2)} = 0 \tag{1.7}$$

[4] We will always assume that the standard technical condition of 'maximal rank' is satisfied [19]; note that this has recently been relaxed [1].

and repeat the argument as before: if the set $S^{(2)}$ of the solutions of this system is not empty and satisfies in addition the condition

$$X^*\Delta^{(2)}\big|_{S^{(2)}} = 0 \tag{1.8}$$

then X is a symmetry for the subset $S^{(2)}$ of solutions of the initial problem (1.1), exactly as before.

Clearly, the procedure can be iterated, and we can say:

Proposition 1. *Given the general differential problem (1.1) and a vector field (1.2), define, with* $\Delta^{(0)} := \Delta$,

$$\Delta^{(r+1)} := X^*\Delta^{(r)}. \tag{1.9}$$

Denote by $S^{(r)}$ *the set of the simultaneous solutions of the system*

$$\Delta^{(0)} = \Delta^{(1)} = \cdots = \Delta^{(r)} = 0 \tag{1.10}$$

and assume that this is not empty for $r \leqslant s$. *Assume moreover that*

$$\begin{aligned} &X^*\Delta^{(r)}\big|_{S^{(r)}} \neq 0 \qquad \textit{for} \quad r = 0, 1, \ldots, s-1 \\ &X^*\Delta^{(s)}\big|_{S^{(s)}} = 0. \end{aligned} \tag{1.11}$$

Then the set $S^{(s)}$ *provides a family of solutions to the initial problem (1.1) which is mapped into itself by the transformations generated by* X.

We shall say that X is a 'partial symmetry', or P-symmetry for short, for the problem (1.1), and that the globally invariant subset of solutions $\mathcal{P} := S^{(s)}$ obtained in this way is a 'X-symmetric set'. We also refer to the number s appearing in the statement as the order of the P-symmetry.

It is clear that, given a differential problem and a vector field X, it can happen that the above procedure gives no result, i.e. that at some pth step the set $S^{(p)}$ turns out to be empty. Just to give an example, consider, with $\ell = 1$, $q = 1$, $p = 2$ and putting $x_1 = x$, $x_2 = y$, the PDE

$$xu_x + x^2u_y + 1 = 0 \tag{1.12}$$

and the vector field, generating the translations along the variable x,

$$X = \frac{\partial}{\partial x}. \tag{1.13}$$

It is easy to verify that, after two steps of the above procedure, one obtains an inconsistent condition. Then, in this case, we simply conclude that (1.13) is not a P-symmetry for the problem (1.12).

Assume instead that a vector field X is a P-symmetry for a given problem, and therefore that a non-empty set $S^{(s)}$ of X-symmetric solutions has been found. We stress that the solutions in this set are, in general, *not* X-invariant: only the set $S^{(s)}$ is globally invariant, while the solutions are transformed into one another under the X action (see the simple example reported in the introduction, and the examples in sections 5 and 6). Note that when there is some solution u_0 which is *invariant* under a given P-symmetry X, then X is also a *conditional symmetry* for the differential problem at hand (see the next section for a discussion of this point). The set of solutions in $S^{(s)}$ will be constituted by one or more *orbits* under the action of the one-parameter Lie group $\exp(\lambda X)$, and (apart from the trivial case of the X-invariant solutions) each one of these orbits can be naturally parametrized by the real Lie parameter λ. Denoting by $u^{[\lambda]} := u(x;\lambda)$ the solutions belonging to any given orbit, one has that each family $u^{[\lambda]}$ satisfies the differential equation (in 'evolutionary form' [19])

$$Qu^{[\lambda]} = \frac{\mathrm{d}u^{[\lambda]}}{\mathrm{d}\lambda} \tag{1.14}$$

where

$$Q = -\xi_i \frac{\partial}{\partial x_i} + \varphi_\alpha \frac{\partial}{\partial u_a}. \tag{1.15}$$

Remark 1. It should be recalled [19] that when we consider ordinary Lie-point symmetries and pass from the infinitesimal (Lie algebra) level to the finite (Lie group) one, in general we have only a *local* Lie group, i.e. the map $T_\lambda = \mathrm{e}^{\lambda X}$ is a symmetry of the differential equation only for λ in some interval $|\lambda| < c_0$. Clearly, the same will apply here.

Remark 2. In the determination of P-symmetries and of orbits of solutions, the 'higher' equations $\Delta^{(r)}$ (with $r \neq 0$) in the hierarchy will only be considered on the submanifold $S^{(p)}$, $p \leqslant r$; thus we can also consider them directly on $S^{(r)}$ from their introduction (and further restrict them if the procedure has to go on). This has no conceptual advantage, but can be appropriate for computational ease, as we will see below in the examples.

Remark 3. Our above procedure can be given a nice geometrical interpretation; to discuss this, we focus on the finite action of the vector field X (1.2), or more precisely of its prolongation X^*, on the differential operator (1.1). This is given by

$$\mathrm{e}^{\lambda X^*}\Delta = \Delta + \lambda X^*\Delta + \frac{\lambda^2}{2!}(X^*)^2\Delta + \cdots = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}(X^*)^k\Delta \tag{1.16}$$

where λ is the Lie parameter. If X is an exact symmetry of Δ, this must be zero whenever $\Delta = 0$; indeed, we know that with X a symmetry, $X^*\Delta = 0$ on the solution set $S^{(0)}$, and therefore *a fortiori* $(X^*)^k\Delta = 0$ on this same set.

Now we note that with our construction, $\Delta^{(0)} := \Delta$, $\Delta^{(1)} = X^*(\Delta)$, $\Delta^{(2)} = X^*\big(X^*(\Delta)\big)$, and so on; our condition that X^* is a symmetry for the solution set $S^{(s)}$ is then

$$(X^*)^s\Delta\big|_{S^{(s)}} = 0. \tag{1.17}$$

Looking back at (1.16), we rewrite it in the form

$$\mathrm{e}^{\lambda X^*}\Delta^{(0)} = \Delta^{(0)} + \sum_{r=1}^{s-1} \frac{\lambda^r}{r!}\Delta^{(r)} + \sum_{k=s}^{\infty} \frac{\lambda^k}{k!}(X^*)^k\Delta^{(0)}. \tag{1.18}$$

Now, the conditions (1.10) on the chain of equations $\Delta^{(r)} = 0, r = 0, 1, \ldots, s-1$, together with (1.11) or (1.17) show that the right-hand side of (1.18) does actually vanish on the solution set $S^{(s)}$. In other words, the fact that X is a partial symmetry for $\Delta^{(0)} = 0$ guarantees that the second sum in (1.18) vanishes; requiring by hand the vanishing of each term in the first sum guarantees that the whole series vanishes, and thus identifies a set of conditions sufficient to guarantee that each solution of $\Delta^{(0)} = 0$ complying with these conditions is transformed into another—in general different—solution of $\Delta^{(0)} = 0$ complying with the same conditions.

Remark 4. Our discussion is in terms of standard Lie-point symmetries; it is well known that one can also introduce *strong* symmetries [13, 19], i.e. those for which $X^*(\Delta) = 0$ on all of the jet space (and not just on the solution manifold $\mathcal{S}_\Delta$, i.e. not just when $\Delta = 0$); the relation between standard and strong symmetries has been clarified in [5]. It should be quite clear that our approach could also be reformulated in terms of strong symmetries and 'strong P-symmetries'; these would be defined by lifting the restriction to relevant solution manifolds and sets (e.g. in (1.5), (1.8) and (1.11)). We will not discuss this setting, but note that in this case one would still obtain a set of solutions to the original equation which is globally X-invariant (we recall a strong symmetry is also a symmetry), but the occurrence of such a set would be even rarer, as X would be required to be a strong symmetry of the system (1.10). On the other hand, computations would be more straightforward, as one would not have to perform the substitutions needed to implement restrictions to solution sets.

Remark 5. After the completion of this paper, one of the referees pointed out that our approach is related to the geometrical method of Gardner [14, 17], which is itself related to Cartan's ideas, and in particular with the concept of the k-stable vector fields. The relation between partial symmetries and Gardner's approach appears to be not trivial. We do not discuss this relationship here, but just mention that our focus is on symmetry properties, while Gardner's one is essentially on geometrical structures.

2. Partial symmetries and conditional symmetries

It should be noted that the procedure presented here is related to, but quite different in spirit from, the standard *conditional symmetries* approach in several ways.

Let us briefly recall what conditional symmetries are and how they are determined (see [18, 32–34] for a more complete discussion). Given an mth-order differential equation $\Delta(x, u^{(m)}) = 0$, we say that

$$X = \xi_i \frac{\partial}{\partial x_i} + \varphi_\alpha \frac{\partial}{\partial u_\alpha} \qquad (i = 1, \ldots, p; \alpha = 1, \ldots, q) \tag{2.1}$$

is a conditional symmetry for Δ if there is some solution $u(x)$ to $\Delta = 0$ which is X-invariant. The X-invariance condition can be written as

$$\varphi_\alpha(x, u) - \sum_{i=1}^{p} \xi_i(x, u) \frac{\partial u_\alpha}{\partial x_i} = 0 \qquad (\alpha = 1, \ldots, q) \tag{2.2}$$

so that X-invariant solutions to $\Delta = 0$ are also solutions to the system

$$\begin{aligned} &\Delta(x, u^{(m)}) = 0 \\ &\varphi_\alpha(x, u) - \xi_i(x, u)(\partial u_\alpha/\partial x_i) = 0 \qquad \alpha = 1, \ldots, q. \end{aligned} \tag{2.3}$$

By construction, X will be an ordinary Lie-point symmetry for the system (2.3). Note that $X^{(1)}[\varphi_\alpha - \xi_i(\partial u_\alpha/\partial x_i)] \equiv 0$, by construction, where $X^{(1)} := \mathtt{pr}^{(1)} X$ is the first prolongation of X, so that standard Lie-point symmetries (i.e. their generators Y) of the system (2.3) are computed by just requiring that

$$Y^{(m)}[\Delta]_{S^*} = 0 \tag{2.4}$$

where $Y^{(m)} := \mathtt{pr}^{(m)} Y$ and S^* is the solution manifold for the system (2.3), and thus corresponds to X-invariant solutions to $\Delta = 0$.

Note that these symmetries will leave globally invariant the set of X-symmetric solutions to $\Delta = 0$; the special symmetry $Y = X$ will leave each of these solutions invariant. If we look for solutions which are invariant under a different vector field X, the system (2.3) would also be changed; thus, conditional symmetries do not have any reason to form a Lie algebra. We also recall that the determining equations for conditional symmetries are nonlinear (see [18, 32–34]).

After having so sketchily recalled the basic notions about conditional symmetries, let us comment on the relations and differences between these and the partial symmetries (defined above) we are discussing in this paper.

Let us first comment on similarities. Comparing the definitions of conditional and partial symmetries, it is clear that, as already remarked, *any conditional symmetry for* Δ *is also a partial symmetry for* Δ. Indeed, if there exists a solution $u_0(x)$ to $\Delta = 0$ which is X-invariant, we are guaranteed of the existence of a non-empty subset S_X of the solution set $\mathcal{S}_\Delta$ which is globally X-invariant; in the worst case S_X consists of the solution u_0 alone: in this case we should consider the partial symmetry to be trivial.

We also note that the system (1.10) does obviously (by construction) admit X as a standard Lie-point symmetry; this is similar to what happens for conditional symmetries (see, e.g., [34]).

Let us now comment on differences between conditional and partial symmetries. First, we note that for partial symmetries we do not require *invariance* of any solution under the vector field, but only global invariance of a family of solutions; indeed, here we are looking for solutions of (1.1) also satisfying (1.14) and (1.15), but not necessarily such that (2.2) are satisfied.

Second, in the standard conditional symmetry approach, the equation $\Delta = 0$ is supplemented with a side condition (i.e. (2.2) introduced above) which, as just remarked, is different for different vector fields (as also happens for partial symmetries) but which is independent of the differential operator Δ in consideration; here instead the conditions depend not only on the vector field X but also on the equation which we are studying and which gives rise to the hierarchy of equations $\Delta^{(r)} = 0$. Thus, on the one hand, we aim to identify partial symmetries and through these to identify sets of solutions which are more general than in the conditional symmetries approach; on the other hand, the tools we are using for this are more specific to the single equation to be considered.

It should also be noted that in searching for ordinary Lie-point symmetries, the determining equations for the unknown ξ_i, φ_α are necessarily linear. In the conditional symmetries approach, one supplements the differential system $\Delta = 0$ with a linear equation, expressing the invariance of the solution $u(x)$ under an undetermined vector field X with coefficients ξ_i, φ_α; as a consequence of the double role of the φ, ξ, the determining equations for the conditional symmetries, i.e. for φ, ξ, are nonlinear. For partial symmetries, we supplement $\Delta = 0$ with conditions which depend on both φ, ξ and Δ itself, and the $\Delta^{(r)}$ could depend in general on products of r coefficients φ, ξ; if we think of determining all the possible partial symmetries (say of given order) of an equation $\Delta = 0$, the determining equations for these are again and unavoidably nonlinear.

This also means that we have no systematic way of solving or attacking them, and actually in general we have no hope of finding even a special solution, i.e. a single partial symmetry, of a given equation. This should not be a surprise, as: (a) partial symmetries are a non-generic feature of differential equations and (b) conditional symmetries are a special case of partial symmetries (see above), and we have no algorithmic way of completely determining the conditional symmetries of a given equation.

Thus the present method is expected to be relevant mainly when we either have some hint (e.g. on a physical basis) of what partial symmetries could be, or we are especially interested in specific candidates for partial symmetries (again, for example, on the basis of physical relevance) and want to investigate if this is the case and to determine the set $S^{(s)}$ of solutions which is globally invariant under these partial symmetries.

Needless to say, the fact that conditional symmetries are also partial symmetries (but the set of X-invariant solutions is in general only a subset of the maximal X-symmetric set of solutions) provides natural candidates for the search of non-trivial partial symmetries.

Remark 6. Rather than looking for partial symmetries of a given equation, one could be interested in the dual problem: determining all the differential equations (say, with given dimensionality of the dependent and independent variables and of given order) which admit a given vector field X as a partial symmetry (say, of given order s). This is a hard problem, mainly because of the substitutions to implement in order to restrict to appropriate solution sets and manifolds: these make the determining equations nonlinear in Δ. However, if we require X to be a strong partial symmetry (see remark 4 above) this problem is not present, and equations (1.9) and (1.11) are linear as equations for Δ.

Let us also stress that (as already mentioned in the introduction) our notion of P-symmetry can be extended immediately, repeating word for word the above procedure, to the case of generalized (or Bäcklund) symmetries. The only difference is that here one considers vector fields of the form [19]

$$X = \varphi_\alpha(x, u, u^{(1)}, u^{(2)}, \ldots)\frac{\partial}{\partial u_\alpha} \tag{2.5}$$

where φ_α also depend on the derivatives $u_{\alpha,i}, u_{\alpha,ij}, \ldots$, with $u_{\alpha,i} = \partial u_\alpha/\partial x_i$, etc (denoted globally by $u^{(1)}, u^{(2)}, \ldots$).

The case of conditional Bäcklund symmetries has been considered by Zhdanov (see [35]), where some physically interesting examples are also provided. However, exactly as in the case of standard Lie-point symmetries, our notion of partial Bäcklund symmetries is different from (and actually—in some cases—extends) the notion of conditional Bäcklund symmetries. Example 5 in section 5 below (which is a modification of an example presented in Zhdanov's paper [35]), although quite simple, will show, in fact, that a nonlinear PDE may possess a P-Bäcklund symmetry X, and therefore may possess an X-symmetric family of solutions (i.e. a family of solutions such that the partial Bäcklund symmetry maps any solution of this family into another of the same family, just as in the case of partial Lie-point symmetries), but which are *not* invariant under this X: this implies that X is not a conditional Bäcklund symmetry for the given PDE.

3. Partial superposition principle

We will consider here a special situation, which can be naturally included in the above notion of P-symmetry.

Consider vector fields of the form

$$X = \varphi_\alpha(x)\frac{\partial}{\partial u_\alpha}. \tag{3.1}$$

It may be interesting to note that, given a differential problem $\Delta = 0$, applying X^* to Δ is in this case nothing but evaluating the Fréchet derivative of Δ applied to the vector function $\varphi_\alpha(x)$:

$$X^*\Delta = \frac{\partial\Delta}{\partial u_\alpha}\varphi_\alpha(x) + \frac{\partial\Delta}{\partial u_{\alpha,i}}\varphi_{\alpha,i}(x) + \frac{\partial\Delta}{\partial u_{\alpha,ij}}\varphi_{\alpha,ij}(x) + \cdots =: \mathcal{L}(u, \Delta)\varphi \tag{3.2}$$

where $\mathcal{L}(u, \Delta)$ is a linear operator. If the transformation generated by (3.1) is an exact symmetry, this implies that, given any solution $u_0(x)$ of $\Delta = 0$, then also $u = u_0(x) + \lambda\varphi(x)$ is a solution. However, as in the previous cases, it can happen that the symmetry condition $\mathcal{L}(u, \Delta)\varphi\big|_{\Delta=0} = 0$ is not satisfied in general, but only by some subset of solutions: $\mathcal{L}(u, \Delta)\big|_{S^{(s)}}\varphi = 0$. This means that $u_0(x) + \lambda\varphi(x)$ may be a solution to $\Delta = 0$ only for some special $u_0(x)$ (and $\varphi(x)$). This gives rise to a sort of 'partial superposition principle' for nonlinear equations. For instance, if

$$\Delta := u_x + u - 1 + u_x^2(u_x - u_y) = 0 \tag{3.3}$$

and choosing $\varphi = \exp(-x - y)$, one can easily verify that

$$u^{[\lambda]}(x, y) = 1 + \lambda \exp(-x - y) \tag{3.4}$$

is an X-symmetric family of solutions to (3.3) for any λ.

The case of vector fields of the form

$$X = \varphi_\alpha(u)\frac{\partial}{\partial u_\alpha} \tag{3.5}$$

is similar to the previous one (3.1), and will be considered in the next section in the special context of dynamical systems.

Remark 7. It is known that there are classes of equations admitting *nonlinear* superposition principles [6, 23, 29]; it has to be expected that this construction could extend to this setting, leading to 'partial nonlinear superposition principles', but such a discussion would go way beyond the scope of this paper. For an extension of the linear superposition principle, see also [27].

4. Dynamical systems

Some attention must be reserved for the special but important case of dynamical systems, i.e. of systems of differential equations of the form

$$\dot{u} = f(u) \tag{4.1}$$

where the independent variable is the time t and $u_\alpha = u_\alpha(t) \in \mathbb{R}^n$ ($p = 1, \ell = q = n$). Here, $\dot{u} = \mathrm{d}u/\mathrm{d}t$, $f = f(u)$ is assumed to be a smooth vector-valued function (we consider for simplicity autonomous problems, in which f is independent of time).

As is well known, a vector field

$$X = \varphi_\alpha\frac{\partial}{\partial u_\alpha} \qquad \text{with} \quad \varphi_\alpha = \varphi_\alpha(u) \tag{4.2}$$

is a Lie-point (time-independent) exact symmetry of (4.1) if and only if

$$\left[f_\alpha\frac{\partial}{\partial u_\alpha}, \varphi_\beta\frac{\partial}{\partial u_\beta}\right] = 0 \tag{4.3}$$

which expresses just the condition $X^*\Delta\big|_{\Delta=0} = 0$, with $\Delta := (\mathrm{d}u/\mathrm{d}t) - f(u)$. If this commutator is not zero, the first condition $\Delta^{(1)} = 0$, i.e.

$$\psi_\alpha(u) := f_\beta\frac{\partial\varphi_\alpha}{\partial u_\beta} - \varphi_\beta\frac{\partial f_\alpha}{\partial u_\beta} \equiv (f\cdot\nabla_u)\varphi_\alpha - (\varphi\cdot\nabla_u)f_\alpha = 0 \tag{4.4}$$

becomes, once $f(u)$ and $\varphi(u)$ are given, a system of conditions for the u_α, which determines a subset (if not empty) in $\mathbb{R}^n$. In this situation, it may be quite easy to verify directly if this set contains (or otherwise) an X-symmetric family of solutions $u^{[\lambda]}(t)$.

There are some interesting and physically relevant cases in which this situation actually occurs. Note, for instance, that if one of the solutions, say $u^{[0]} = u^{[0]}(t)$, of an X-symmetric family exhibits a well defined time behaviour (e.g. it is periodic with some period T, or it is a homoclinic or heteroclinic orbit), and if the vector field X is defined globally, i.e. along all of the time trajectory of $u^{[0]}(t)$, then all the solutions of the family $u^{[\lambda]}(t)$ exhibit the same time behaviour [13].

Given a family $u^{[\lambda]} = u^{[\lambda]}(t)$ of solutions to (4.1), and denoting by $\mathcal{L}_{[\lambda]}(f)$ the linearization of $f(u)$ evaluated along $u^{[\lambda]}$, i.e.

$$\mathcal{L}_{[\lambda]}(f) := \nabla_u f\big|_{u^{[\lambda]}} \tag{4.5}$$

it can be useful—in view of its applications (see below)—to state the above argument in the following form:

Proposition 2. *Assume that the dynamical system (4.1) admits a partial symmetry X of the form (4.2) and let $u^{[\lambda]} = u^{[\lambda]}(t)$ be an orbit of solutions obtained under the action of the group generated by X. Then $\varphi = \varphi(u^{[\lambda]})$ satisfies the equations*

$$\dot{\varphi} = \mathcal{L}_{[\lambda]}(f) \cdot \varphi \tag{4.6}$$

and

$$\varphi = \frac{\mathrm{d}u^{[\lambda]}}{\mathrm{d}\lambda}. \tag{4.7}$$

In order to prove this, just note that $\dot{\varphi} = \dot{u} \cdot \nabla_u \varphi = f \cdot \nabla_u \varphi$, then (4.6) and (4.7) come from (4.4) and from equations (1.14) and (1.15).

Remark 8. This proposition is relevant, for example, in the cases where the dynamical system admits a manifold of homoclinic (or heteroclinic) orbits. Indeed, proposition 2 ensures that the vector $\varphi = \mathrm{d}u^{[\lambda]}/\mathrm{d}\lambda$, tangent to the family $u^{[\lambda]}$, is a bounded solution (for all $t \in \mathbb{R}$) of equation (4.6), which is usually called the 'variational equation', obtained by linearizing the dynamical system along $u^{[\lambda]}$. On the other hand, knowledge of all bounded solutions to the variational equation is important to construct the Mel'nikov vector, which provides a useful tool for determining the onset of chaotic behaviour in the dynamical system in the presence of perturbations. For the applications of this fact to the theory of chaotic behaviour of a dynamical system, which clearly goes beyond the scope of this paper, we refer, for example, to [15, 16, 28].

Remark 9. In the same context as in remark 8, it is well known that another bounded solution of the variational equation (4.6) is provided by the time derivative $\mathrm{d}u/\mathrm{d}t$; it can be remarked that, from the group-theoretical point of view, the two tangent vectors $\mathrm{d}u/\mathrm{d}\lambda$ and $\mathrm{d}u/\mathrm{d}t$ are to be considered perfectly on the same level, indeed the time evolution of the dynamical system is always a (nonlinear) symmetry for the system, with generator

$$f_\alpha \frac{\partial}{\partial u_\alpha} = \frac{\mathrm{d}}{\mathrm{d}t} \tag{4.8}$$

where the time t plays the role of the parameter λ.

5. Examples I: PDEs

As pointed out in section 1, our procedure may be fruitful if one is able to conjecture some 'reasonable candidate' for such a partial symmetry. As mentioned before, one may consider first of all the conditional symmetries: examples 1 and 4 will cover situations where, in fact, one finds X-symmetric sets of solutions which contain as a special case an X-invariant solution (which could also be obtained via the standard method of conditional symmetries). Examples 2 and 3, instead, will show cases where the partial symmetry is *not* a conditional symmetry, and, in fact, we are able, by introducing suitable P-symmetries, to obtain X-symmetric sets of solutions which do not contain any X-invariant solution. The same applies to example 5, dealing with generalized symmetries.

Another typical situation which may suggest possible candidates as P-symmetries occurs for instance, as illustrated by the examples 1 and 3 below, when the differential problem is written as a sum of two terms, the first one possessing a known group of exact symmetries, plus a 'perturbation' term which breaks these symmetries. Then the natural candidates are just the symmetries of the unperturbed term.

Other convenient situations may occasionally occur when imposing the chain of conditions $(X^*)^r\Delta^{(0)} = 0$ leads to a simpler differential problem (e.g. thanks to the vanishing of some term in the equation, as in examples 1–3 below). Also, observing that in our procedure each subset $S^{(r)}$ is, in general, considerably smaller than the preceding sets in the chain $S^{(0)} \subset \cdots \subset S^{(r)}$, it can happen that, after just one or very few steps, one is able to 'isolate' directly 'by inspection' within some set $S^{(r)}$ (although $X^*\Delta^{(r)}\big|_{S^{(r)}} \neq 0$) some X-symmetric family of solutions (see example 1).

Example 1 (Modified Laplace equation). Consider the PDE, putting $x_1 = x, x_2 = y$, and with $\ell = 1, q = 1, m = 3$,

$$\Delta := u_{xx} + u_{yy} + g(u)u_{xxx} = 0 \tag{5.1}$$

and the vector field, generating the rotations in the plane x–y,

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \tag{5.2}$$

which is not a symmetry for (5.1) unless $g(u) = 0$ (we shall show below that it is *not* enough to impose $u_{xxx} = 0$). The first application of the prolonged symmetry X^* to (5.1) gives

$$\Delta^{(1)} := X^*\Delta\big|_{\Delta=0} = 3g(u)u_{xxy} \tag{5.3}$$

and then, excluding constant solutions u_0 satisfying $g(u_0) = 0$, we obtain the first condition

$$u_{xxy} = 0. \tag{5.4}$$

Applying the convenient prolongation X^* to this equation gives

$$\Delta^{(2)} = 2u_{xyy} - u_{xxx} \tag{5.5}$$

which does *not* satisfy $\Delta^{(2)}\big|_{S^{(1)}} = 0$, and therefore one obtains the other condition $2u_{xyy} - u_{xxx} = 0$. Iterating the procedure, one has that

$$\Delta^{(3)} = X^*\Delta^{(2)} = -7u_{xxy} + 2u_{yyy} \tag{5.6}$$

which again is not zero on the set $S^{(2)}$, but, now using (5.4), gives the new condition

$$u_{yyy} = 0. \tag{5.7}$$

Then, other steps are necessary; proceeding further, we obtain $X^* u_{yyy} = -3u_{xyy}$, giving $u_{xyy} = 0$ and, finally, from another application of X^* to this, we obtain

$$X^* u_{xyy} = -2u_{xxy} + u_{yyy} = 0 \tag{5.8}$$

which is, in fact, zero, thanks to (5.4) and to (5.7).

Then, in conclusion, equation (5.2) is a P-symmetry of order $s = 5$ for equation (5.1).

The set $S^{(s)}$ of simultaneous solutions of all the above conditions has the general form

$$S^{(s)} = \{u(x, y) = A(x^2 - y^2) + Bxy + Cx + Dy + E\} \tag{5.9}$$

where $A, \ldots, E$ are arbitrary constants, and it is easy to recognize, by putting, for example, $A = a \cos 2\lambda$, $B = a \sin 2\lambda$ and respectively $C = c \cos \lambda$, $D = c \sin \lambda$, that this set contains, apart from the constant solutions $u(x, y) = E$, which are clearly rotationally invariant, two different families of orbits of solutions to the initial problem (5.1), which, in fact, are transformed into themselves under rotations.

It is easy to see that the rotation symmetry X is not only a P-symmetry but also a conditional symmetry for the problem (5.1), indeed, the rotationally invariant solution must be of the form $u = v(\rho)$ with $\rho = (x^2 + y^2)/2$; substituting into (5.1), we obtain

$$2\rho v'' + v' + g(v)(3xv'' + x^3 v''') = 0 \tag{5.10}$$

which (for $g(v) \neq 0$) implies $v' = 0$. Thus the only rotationally invariant solutions are in this case given trivially by the constant ones, which are, in fact, included in the larger X-symmetric set of the solutions (5.9) found above.

The result (5.9) looks quite obvious and indeed could be expected just after one step (or perhaps immediately); this example, however, can be useful for several reasons. First of all, it shows that it could be possible to reach the conclusion by means of an iterative procedure. Second, it also shows that it is *not* sufficient to impose, together with (5.1), only the condition of the vanishing of the 'symmetry-breaking' term

$$u_{xxx} = 0. \tag{5.11}$$

Indeed, this equation does *not* admit the rotation symmetry, therefore a simultaneous solution of both (5.1) and (5.11), e.g. $u = x^2 y - y^3/3$, would be transformed by X into a solution of $u_{xx} + u_{yy} = 0$ but *neither* of (5.1) *nor* of (5.11). Similarly, it is not sufficient to impose that the solutions of the initial equation (5.1) satisfy only the first condition (5.4); e.g. with $g(u) = 1$, the solution $u(x, y) = \exp x$ of (5.1) also satisfies (5.4) but does not belong to any family of solutions of (5.1) which is also globally invariant under rotations.

Example 2 (KdV equation). Consider, with $x_1 = x$, $x_2 = t$, the classical Korteweg–de Vries equation

$$\Delta := u_t + u_{xxx} + u u_x = 0. \tag{5.12}$$

It is well known that it admits an exact scaling symmetry, given by

$$X = -2u \frac{\partial}{\partial u} + x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t}. \tag{5.13}$$

We now want to determine whether there are scaling P-symmetries for the KdV.

We consider the generic scaling vector fields

$$X = au \frac{\partial}{\partial u} + bx \frac{\partial}{\partial x} + ct \frac{\partial}{\partial t} \tag{5.14}$$

(note that $X = bX_0$, where X_0 is the exact symmetry (5.13), for $a = -2b$, $c = 3b$). Applying the (third) prolongation X^* of X on Δ, we obtain

$$\begin{aligned} X^*\Delta := \Delta^{(1)} &= a\Delta + [-cu_t + auu_x - buu_x - 3bu_{xxx}] \\ &= (a-c)\Delta + [(a-b+c)uu_x - (3b-c)u_{xxx} \end{aligned} \tag{5.15}$$

requiring to be on $S^{(0)}$, i.e. on $\Delta = 0$ (the solution manifold of the KdV), i.e. performing the substitution $u_t \to -(u_{xxx} + uu_x)$, we obtain the condition

$$\widetilde{\Delta}_1 := \Delta^{(1)}\big|_{S^{(0)}} = (a-b+c)uu_x - (3b-c)u_{xxx} = 0 \tag{5.16}$$

note that this is identically zero only in the 'trivial' case $X = bX_0$, where X_0 is the exact symmetry (5.13). We rewrite this as

$$\widetilde{\Delta}_1 = Auu_x - Bu_{xxx} = 0 \tag{5.17}$$

and consider different cases.

Case I. If $A = B = 0$ we are, as already remarked, in the case $X = bX_0$ and we are thus considering the case of exact scaling symmetry.

Case II. If $A \neq 0$ and $B = 0$, $\widetilde{\Delta}_1 = 0$ reduces to $uu_x = 0$, which in turn implies $u(x,t) = \alpha(t)$ and, due to $\Delta = 0$, $u(x,t) = c_0$. We reduce then to the trivial case of constant solutions (these are obviously transformed among themselves under the action of any X of the form (5.14), and are invariant under any field with $a = 0$ for the solution with $c_0 \neq 0$, and under any X when $c_0 = 0$).

Case III. If $A = 0$ and $B \neq 0$, then $\widetilde{\Delta}_1 = 0$ reduces to $u_{xxx} = 0$; note that at next step we have $X^*(\widetilde{\Delta}_1) = (a - 3b)u_{xxx}$ which is obviously zero on $S^{(1)}$. The solution set for $u_{xxx} = 0$ corresponds to $u(x,t) = \alpha(t) + \beta(t)x + \gamma(t)x^2$; substituting this into the KdV equation we obtain that $\gamma(t) = 0$ and that

$$\alpha' + \alpha\beta = 0 \qquad \beta' + \beta^2 = 0. \tag{5.18}$$

The second of these yields $\beta(t) = (c_1+t)^{-1}$ and using this we also obtain $\alpha(t) = c_2(c_1+t)^{-1}$. Thus, X is a P-symmetry and the set of solutions to the KdV which is globally invariant under X, with $A = 0$ and $B \neq 0$, is given by

$$u(x,t) = \frac{c_2 + x}{c_1 + t} \tag{5.19}$$

with c_1 and c_2 arbitrary constants. Note that no solution of this form is invariant under such X: thus, these P-symmetries X do *not* correspond to conditional symmetries. As two different examples of this case, one can consider in (5.14) $a = 0, b = c = 1$, which generates the simultaneous dilations of the independent variables x and t (and then, in terms of the Lie parameter λ, under the action of X (cf (1.14) and (1.15)), one has in (5.19) $c_2 = c\exp\lambda, c_1 = c\exp\lambda$, with arbitrary c); or $b = 0, a = -1, c = 1$ which generates the simultaneous dilation of the independent variable t and shrinking of the dependent variable u (and then $c_1 = \exp\lambda, c_2 = c$).

Case IV. If $A \neq 0$ and $B \neq 0$, we have a more interesting case; now $\widetilde{\Delta}_1 = 0$ reads

$$u_{xxx} = \frac{a-b+c}{3b-c}uu_x. \tag{5.20}$$

Applying X^* on $\Delta^{(1)}$, we obtain

$$\begin{aligned} \Delta^{(2)} &:= (a-c)^2\Delta + (a-c)\widetilde{\Delta}_1 + A(2a-b)uu_x - B(a-3b)u_{xxx} \\ &= (a-c)^2\Delta + (3a-b-c)\widetilde{\Delta}_1 + B(a+2b)u_{xxx} \end{aligned} \tag{5.21}$$

when we impose (5.16) and (5.20), this reduces to

$$\widetilde{\Delta}_2 := \Delta^{(2)}\big|_{S^{(1)}} = (a+2b)uu_x = 0. \tag{5.22}$$

If $a+2b \neq 0$, this requires $u_x = 0$, i.e. we are reduced to the case of trivial constant solutions. On the other hand, if

$$a + 2b = 0 \tag{5.23}$$

which also implies $A = B$, we have obtained that X is a P-symmetry of the KdV.

It should be noted that now, thanks to $A = B$, the equation $\widetilde{\Delta}_1 = 0$ reads $u_{xxx} = uu_x$ for all X in this class; this equation can be reduced to 'quadratures', and for functions $u(x,t)$ satisfying this, the KdV reduces to

$$u_t = -2u_{xxx} = -2uu_x = -2u\big[u^3/3 + \alpha(t)u + \beta(t)\big]^{1/2}. \tag{5.24}$$

Note also that in this case we have $X = a[u(\partial/\partial u) - 2x(\partial/\partial x)] + c(\partial/\partial t)$; considering the first two components of X it is sufficient to guarantee that, as can be seen by solving the characteristic equation for X, the only solution invariant under any such X is the trivial one $u(x,t) = 0$.

This completes the possible cases in the analysis of (5.14).

Example 3 (A nonlinear heat equation). Consider this nonlinear heat equation

$$\Delta := u_t - u_{xx} - uu_{xx} + u_x^2 = 0 \tag{5.25}$$

and the vector field

$$X = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} \tag{5.26}$$

(which is an exact symmetry of the standard linear heat equation). One obtains at the first step

$$\Delta^{(1)} = X^*\Delta = x(u_x^2 - uu_{xx}). \tag{5.27}$$

In this case the two conditions

$$\Delta = \Delta^{(1)} = 0 \tag{5.28}$$

are enough to define a set $S^{(1)}$ of solutions which is X-symmetric, indeed, one obtains

$$X^*\Delta^{(1)}\big|_{S^{(1)}} = 0 \tag{5.29}$$

and therefore (5.26) is a P-symmetry of order $s = 1$ for equation (5.25). Equations (5.28) can be easily solved to obtain the X-symmetric family of solutions

$$u^{[\lambda]}(x,t) = c\exp(-x\lambda + t\lambda^2) \tag{5.30}$$

where c is a constant, which is indeed transformed into itself by the finite transformations generated by X, i.e.

$$t \to t' = t \qquad x \to x' = x + 2t\lambda \tag{5.31}$$

$$u \to u^{[\lambda]} = u(x,t)\exp(-x\lambda - t\lambda^2) = u(x' - 2t'\lambda, t')\exp(-x'\lambda + t'\lambda^2). \tag{5.32}$$

We can also verify that the above transformation (5.26) is not a (non-trivial) conditional symmetry for the problem (5.25). Indeed, the functions $v = v(x,t)$ satisfying the invariance condition (2.3) must be of the form

$$v = w(t)\exp(-x^2/4t). \tag{5.33}$$

Inserting in (5.25) gives

$$-2t\frac{w'}{w} = 1 + w\exp(-x^2/4t) \tag{5.34}$$

which can be satisfied only by $w \equiv 0$. This agrees with our previous result (5.30), which shows, in fact, that no solutions of the form (5.33) are included in the family (5.30).

Example 4 (The Boussinesq equation). The Boussinesq equation

$$\Delta := u_{tt} + uu_{xx} + (u_x)^2 + u_{xxxx} = 0 \tag{5.35}$$

has been used as a testing ground for conditional symmetries [9, 18, 33, 34], and thus it is appropriate to (briefly) consider it from the point of view of partial symmetries as well. We consider here only the first (and simplest) one of the conditional symmetries of this equation [18, 33, 34], namely

$$X = \frac{\partial}{\partial t} + t\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial u}. \tag{5.36}$$

Applying our procedure, we find

$$\Delta^{(1)} := X^*\Delta = u_{xt} + tu_{xx} \qquad \Delta^{(2)} := X^*\Delta^{(1)} \equiv 0. \tag{5.37}$$

We then have to look for the simultaneous solutions of the two equations $\Delta = 0$ and $\Delta^{(1)} = 0$. The set $S^{(1)}$ of these X-symmetric solutions is not empty, in fact, it must contain at least the X-invariant solutions to (5.35), which can be obtained via the conditional symmetries approach [18, 33, 34]; actually, we shall see that, as in example 1, this set is much larger. Solving the first condition $\Delta^{(1)} = 0$ gives indeed

$$u(x, t) = w(x - t^2/2) + g(t) \tag{5.38}$$

where w and g are arbitrary; note that only if $g = -t^2$ this solution is invariant under X. We then put for convenience

$$g(t) = -t^2 + h(t). \tag{5.39}$$

Inserting this into the Boussinesq equation, we find that w and h must satisfy

$$\frac{\mathrm{d}}{\mathrm{d}z}(w''' + ww' - w - 2z) + hw'' + \frac{\mathrm{d}^2h}{\mathrm{d}t^2} = 0 \tag{5.40}$$

where $z = x - t^2/2$ and $w = w(z)$. Now, if $h = 0$, the well known equation for $w(z)$ [18, 33, 34] is recovered, but with $h \neq 0$ other solutions of the Boussinesq equation, not invariant under (5.36), can be found. For instance, we obtain the following family of solutions:

$$u(x, t) = w(z) + A \tag{5.41}$$

where A is a constant and $w = w_A(z)$ satisfies the equation

$$w''' + ww' - w + Aw' = c + 2z \tag{5.42}$$

and also the other family of solutions (quite trivial, but not included in the previous set (5.41))

$$u(x, t) = Bx - \tfrac{1}{2}B^2t^2 + Ct + D \tag{5.43}$$

with B, C, D constants. It is clear that all the X-invariant solutions found via the conditional symmetry approach are recovered for particular values of the parameters A, B, C, D.

Example 5 (A PDE admitting a P-Bäcklund symmetry). Finally, we deal with an example of the extension of partial symmetries to Bäcklund symmetries, mentioned at the end of section 2. Consider, as in example 4 of Zhdanov's paper [35], a PDE of the form

$$u_t = u_{xx} + R(u, u_x) \tag{5.44}$$

and the Bäcklund vector field

$$X = (u_{xx} - au)\frac{\partial}{\partial u} \qquad (a \in \mathbb{R}). \tag{5.45}$$

According to the prescriptions for the conditional symmetries, Zhdanov looks for solutions of (5.44) restricted to the manifold of the *invariant* solutions under the transformations generated by the vector field (5.45), i.e. of the solutions also satisfying

$$u_{xx} - au = 0 \tag{5.46}$$

(and its differential consequences), and concludes that (5.45) is a conditional Bäcklund symmetry for (5.44) if and only if the nonlinear term R in (5.44) has a special form, see [35].

We choose instead, as an example,

$$R = u_x^2 - \tfrac{1}{2}au^2 \tag{5.47}$$

which does *not* have the above form, and, obviously, we do *not* impose invariance under X. Applying the prolongation X^* to the PDE (5.44) with a generic R, one finds

$$X^*\Delta = -aR + auR_u + au_xR_{u_x} + R_{uu}u_x^2 + 2R_{uu_x}u_xu_{xx} + R_{u_xu_x}u_{xx}^2 \tag{5.48}$$

where $R_u = \partial R/\partial u$, etc; with our choice (5.47), this gives

$$X^*\Delta = 2\left(u_{xx}^2 - \tfrac{1}{4}a^2u^2\right). \tag{5.49}$$

According to our procedure, we look for the solutions of the equation $\Delta^{(1)} := X^*\Delta = 0$, which are given by (let $a > 0$)

$$u_+(t,x) = \varphi_+(t)\exp\left(\sqrt{a/2}x\right) \qquad \text{or} \qquad u_-(t,x) = \varphi_-(t)\exp\left(-\sqrt{a/2}x\right) \tag{5.50}$$

(and which clearly, as expected, do *not* satisfy the X-invariance condition (5.46)). It is now easy to see that our other condition $\Delta^{(2)} := X^*\Delta^{(1)} = 0$ is satisfied once restricted to the solutions (5.50), showing that these constitute an X-symmetric set, although not X-invariant. Finally, taking into account the equation $\Delta^{(0)} = 0$, i.e. equation (5.44), one finds the two families of solutions $u_+(t,x)$ and $u_-(t,x)$ of the PDE (5.44)

$$u_\pm(t,x) = c_\pm \exp\left((a/2)t \pm \sqrt{a/2}x\right) \tag{5.51}$$

(note incidentally that no combination of $u_+(t,x)$ with $u_-(t,x)$ solves the PDE). It can be immediately seen that each one of these families is mapped into itself by the vector field X, and therefore we can conclude that X is a P-Bäcklund symmetry (but not a conditional Bäcklund symmetry) for the PDE (5.44) with the nonlinear term given in (5.47).

6. Examples II: dynamical systems

In this section we briefly consider some cases where the discussion of the above section 4 can be applied.

Example 6. Consider a three-dimensional dynamical system, with $u \in \mathbb{R}^3$, $u := (x, y, z)$, of the form

$$\begin{aligned} \dot{x} &= x(1-r^2) - y + zg_1(x, y, z) \\ \dot{y} &= y(1-r^2) + x + zg_2(x, y, z) \\ \dot{z} &= zg_3(x, y, z) \end{aligned} \tag{6.1}$$

where $g_\alpha(x, y, z)$, $\alpha = 1, 2, 3$ are arbitrary smooth functions and $r^2 = x^2 + y^2$. It is easy to verify that considering the vector field, generating rotations in the plane (x, y),

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \tag{6.2}$$

the partial symmetry condition (4.3) takes the form

$$zG_\alpha(x, y, z) = 0 \qquad \alpha = 1, 2, 3 \tag{6.3}$$

where, for instance, $G_1(x, y, z) = g_2 - y(\partial g_1/\partial x) + x(\partial g_1/\partial y)$, which is non-zero for generic g_α. As is obvious in this simple example, the dynamical system exhibits rotation symmetry once restricted to the plane $z = 0$, and in this plane one can find three different families of solutions $u^{[\lambda]}(t)$ which are mapped into themselves by the rotations: the trajectories lying in $r^2 < 1$, and respectively in $r^2 > 1$, spiralling towards the limit cycle $r^2 = 1$, and the solutions running on the single trajectory which is left fixed by the partial symmetry (the limit cycle).

Example 7. This example is, admittedly, a somewhat artificial one. Indeed, it has been constructed to put together, in a non-symmetric dynamical system, the presence of a partial nonlinear symmetry, and of a two-dimensional heteroclinic manifold. Let us consider then, with $u := (x, y, z) \in \mathbb{R}^3$, the system

$$\begin{aligned} \dot{x} &= x\big(1 - z\exp(-y)\big) + g_1(x, y, z)(R^2 - z)^2 \\ \dot{y} &= y\big(1 - z\exp(-y)\big) + g_2(x, y, z)(R^2 - z)^2 \\ \dot{z} &= -z + yz\big(1 - z\exp(-y)\big) - z^2\exp(-y) + 3R^2 \end{aligned} \tag{6.4}$$

where g_1, g_2 are arbitrary smooth functions, and $R^2 = \frac{1}{6}(x^2 + y^2)\exp(+y) + \frac{1}{2}z^2\exp(-y)$. Note first of all that, if $g_1 = g_2 = 0$, the dynamical system would admit the (exact) nonlinear symmetry

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} - xz\frac{\partial}{\partial z}. \tag{6.5}$$

This symmetry has been introduced in [8], where it has also been shown that the most general dynamical system admitting this symmetry has the form

$$\begin{aligned} \dot{x} &= xf(r^2, v) + yg(r^2, v) \\ \dot{y} &= yf(r^2, v) + yg(r^2, v) \\ \dot{z} &= zh(r^2, v) + yzf(r^2, v) + xzg(r^2, v) \end{aligned} \tag{6.6}$$

where f, g, h are arbitrary functions of the quantities $r^2 = x^2 + y^2$ and $v = z\exp(-y)$. It is not difficult to show that the dynamical system (6.4) possesses a two-dimensional manifold $u^{[\lambda]}$ of heteroclinic orbits, joining biasymptotically the critical points $O = (0, 0, 0)$ and $A = (0, 0, 2)$, and given by

$$u^{[\lambda]}(t, t_0, \lambda) \equiv \left(\sqrt{3}\,\mathrm{sech}(t - t_0)\cos\lambda, \sqrt{3}\,\mathrm{sech}(t - t_0)\sin\lambda,\right.$$
$$\left.\left(1 + \tanh(t - t_0)\right)\exp\left(\sqrt{3}\,\mathrm{sech}(t - t_0)\sin\lambda\right)\right) \tag{6.7}$$

where t_0 is arbitrary. Although the dynamical system (6.4) does not admit in general the symmetry (6.5), we can easily check that this is a partial symmetry for the system (6.4), and, in fact, each one of the heteroclinic orbits in the manifold (6.7), obtained by keeping λ fixed and varying t, is transformed into another orbit of the same manifold by the transformations generated by (6.5). Indeed, the finite action of this transformation on the coordinates is given by

$$\begin{aligned} x &\to x' = x\cos\lambda + y\sin\lambda \\ x &\to y' = -x\sin\lambda + y\cos\lambda \\ z &\to z' = z\exp(-x\sin\lambda + y\cos\lambda) = z\exp(y'). \end{aligned} \tag{6.8}$$

According to remarks 8 and 9 of section 4, we can also directly verify that the two tangent vectors $\mathrm{d}u/\mathrm{d}t$ and $\mathrm{d}u/\mathrm{d}\lambda$ are solutions of the variational equation (4.6) obtained from (6.4).

Acknowledgments

We would like to warmly thank Decio Levi for some useful discussions, right suggestions (in particular, remark 3 is due to him), and encouragement. We also thank Professor P J Olver for a useful (E-mailed) comment, and Professor J F Cariñena for sending us the preprint version of his forthcoming book. Finally, we thank unknown referees for pointing out the relations of our work to a geometrical approach by Gardner (see remark 5), and for encouraging us to explicitly consider generalized symmetries as well.

Appendix. Discrete partial symmetries

It should be noted that the construction and results proposed here, and discussed within the framework of continuous Lie-point transformations, do also apply to more general kinds of transformations, such as non Lie-point ones (see [19]; we have briefly considered here the case of Bäcklund symmetries) and discrete Lie-point transformations. In this appendix we briefly discuss the application of our approach to the latter case.

In this respect, we would like to recall that the main obstacle for the use of discrete symmetries in connection with differential equations is the difficulty in their determination: indeed, except for discrete symmetries which are immediately evident (such as parity transformation or shift by a period) we have no algorithmic way of solving the determining equations for discrete symmetries; this is due to the fact in this case we cannot reduce to the tangent space of suitable manifolds, and thus the determining equations are nonlinear. In the present case, nonlinearity is already present for continuous P-symmetries, and thus determination of possible discrete P-symmetries is a comparably difficult task; as already mentioned in discussing continuous ones, we have some hope of success only if we are led by physical considerations or if we want to analyse (again on a physical basis) a specific kind of transformation. Note, however, that in this respect there are several discrete transformations to be considered, which are natural in physical terms and which are quite interesting if they happen to be P-symmetries: these are reflections and discrete translations. In some contexts, for example, in systems relevant in statistical mechanics [7], one would also be especially interested in discrete scale transformations.

The similarity between the study and determination of discrete and continuous P-symmetries is particularly transparent in terms of the previous remark 3.

We can thus consider a general map $R : (x, u) \to (\widetilde{x}, \widetilde{u})$ and its prolongation R^* acting on $(x, u^{(m)})$; we apply this on the differential equation Δ. If

$$R^*\Delta\Big|_{\Delta=0} = 0 \tag{A.1}$$

then R is a discrete Lie-point exact symmetry of $\Delta = 0$; we assume that (A.1) is not satisfied, and write

$$\Delta^{(1)} := R^*(\Delta). \tag{A.2}$$

We will then consider the common solution set $S^{(1)}$ of Δ and of $\Delta^{(1)}$, and consider $R^*(\Delta^{(1)})$ on this; if this is non-zero, we will iterate the procedure as in the continuous case, until we reach an s such that $R^*\Delta^{(s)}\big|_{S^{(s)}} = 0$. This $S^{(s)}$ identifies a set of solutions to $\Delta = 0$ which is R-symmetric, and our results apply in this setting as well. Note, however, that in this case we cannot iterate our procedure indefinitely if, as it happens for many interesting discrete transformations, there is a $k > 0$ such that $R^k = I$.

Example A.1. Consider the equation

$$\Delta := u_{xx} + u_{yy} + g(u)u_{xxx} = 0 \tag{A.3}$$

and the discrete transformation corresponding to x-reflection,

$$R : (x, y; u) \to (-x, y, u). \tag{A.4}$$

It is easy to see that R^* leaves $g(u)$, u_{xx} and u_{yy} invariant, and maps u_{xxx} in minus itself; thus,

$$\Delta^{(1)} := R^*\Delta = u_{xx} + u_{yy} - g(u)u_{xxx} \tag{A.5}$$

which on $S^{(0)}$ yields

$$\widetilde{\Delta}^{(1)} = -2\, g(u)\, u_{xxx}. \tag{A.6}$$

Therefore, $S^{(1)}$ corresponds to solutions of $\Delta = 0$ satisfying the additional condition $u_{xxx} = 0$; note that with this $\Delta = 0$ reduces to the wave equation $u_{xx} + u_{yy} = 0$ restricted to the space of functions $u(x, y) = \alpha(y) + \beta(y)x + \gamma(y)x^2$. Therefore, we have $\beta''(y) = \gamma''(y) = 0$, and $\alpha''(y) = -2\gamma(y)$ (which in turns implies $\alpha^{iv}(y) = 0$).

Example A.2. Let us consider a system with boundary conditions $u(0, t) = u(2\pi, t) = 0$ and depending on an external constant μ, i.e.

$$\Delta := u_t - \mu u - u_{xx} + u_{xxx} = 0. \tag{A.7}$$

It is easy to see, passing to a Fourier representation, that the solution $u_0(x, t) \equiv 0$ is stable for $\mu < 1$, while for $\mu > 1$ this is unstable and we have instead stable periodic solutions; note that looking for solutions in the form $u(x, t) = f_{k\omega} \exp[\mathrm{i}(kx + \omega t)$ the dispersion relations turn out to be

$$\omega = -\mathrm{i}(\mu - k^2) \tag{A.8}$$

and the boundary conditions impose that k is an integer. The consideration of higher-order terms would allow one to obtain $u(x, t)$ as a Fourier series in terms of x-periodic functions of period 2π and higher harmonics, i.e.

$$u(x, t) = \sum_{k=1}^{\infty} f_k(t) \sin(kx). \tag{A.9}$$

We will consider the discrete transformation corresponding to shift of π in x, i.e.

$$R : (x, t; u) \to (x + \pi, t; u) \tag{A.10}$$

note that this does not in general respect the boundary conditions.

It is easy to see that $R^* \Delta\big|_{\Delta=0} = \mu[u(x, t) - u(x - \pi, t)]$ and thus reduction to $S^{(1)}$ corresponds to

$$u(x, t) - u(x - \pi, t) = 0 \tag{A.11}$$

i.e. to the requirement that only even harmonics are present in the Fourier expansion for $u(x, t)$, that is, in (A.8) all the $f_k(t)$ for odd k are identically zero. Note this means, in particular, that the fundamental wavenumber for $u(x, t)$ will be not 1, but 2.

We remark, although this goes beyond the limits of the present paper, that when μ is not a constant but a varying external control parameter, the problem (A.7) presents a Hopf bifurcation at $\mu = 1$; if we restrict to the subset of solutions $S^{(1)}$, i.e. if we impose the additional boundary condition (A.10), we still have a Hopf bifurcation, but now at $\mu = 4$.

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